Structural Morphology of Polyhedral Spatial Trusses in Static Equilibrium via 4-polytopic Stress Functions

M. Konstantatou[1,*], M. Akbarzadeh[2], A. McRobie[3]

[1] Specialist Modelling Group, Foster + Partners
Riverside, 22 Hester Rd, Battersea, London, United Kingdom

[2] Polyhedral Structures Laboratory, Weitzman School of Design, University of Pennsylvania
Meyerson Hall G19, 210 South 34th Street, Philadelphia, USA

[3] Department of Engineering, University of Cambridge
Trumpington St, Cambridge, United Kingdom

[*] Corresponding author e-mail: mkonstantatou@fosterandpartners.com

Abstract

Graphic statics, a 19th century methodology for the design and analysis of trusses in static equilibrium, has re-emerged recently as a comprehensive framework for designing and analyzing materially efficient structures. Specifically, computational frameworks have been introduced for the structural morphogenesis of compression-only 3D polyhedral trusses by Akbarzadeh et al. (2015). Moreover, the fundamental relation between tension-and-compression 3D trusses in static equilibrium and 4D stress functions has been showed in McRobie (2016). Furthermore, a direct mathematical construction for generating pairs of reciprocal 4D stress functions, and by projection pairs of reciprocal 3D form and force diagrams, has been discussed in Konstantatou et al. (2018). The central role of stress functions in structural morphogenesis was already known by Maxwell (1864a) who used projections of polyhedral (Airy) stress functions to derive 2D trusses in static equilibrium while obtaining design and analysis freedoms. In this paper we apply these constructions to their higher-dimensional equivalents and show how structural morphogenesis of 3D polyhedral, trusses can also have as a starting point 4D stress functions of the form space. The presented direct method has a generative aspect where an underlying grammar can produce a wide range of typologies and can be applied to compression-and-tension cases.

Keywords: Graphic statics, reciprocal diagrams, Airy stress functions, structural morphology, structural design

1 Introduction

Contemporary built environment industry is faced with a lack of effective communication between those who design forms and those who analyse them. Moreover, a need for material efficiency with regards to how we build our structures is more pressing than ever. Thus, there is a growing tendency to shift towards interactive design and analysis tools which can inform the early conceptual design stages with structural performance - leading to more materially efficient structures - whilst adopting a language common to a wide range of researchers and practitioners. Such a language could be the visual and intuitive language of geometry. To this end, a framework of particular importance - which has re-emerged again recently - is the one of graphic statics.

Graphic statics is a 19th century method for the analysis and design of structures in static equilibrium. While they were popular among practitioners in the second half of the 19th century, they saw a 20th century decline; however, in the turn of the 21st century they have re-emerged largely due to significant advances in Computer Aided Design (CAD) and contemporary visualisation capabilities. This is because, CAD developments enabled the generalisation of graphic statics to the 3rd dimension and their implementation in terms of computational design and analysis frameworks for a variety of case studies Graphic statics, and more generally geometry-based analysis and design frameworks, hold the promise of facilitating the bridging of the gap between structural and architectural design, and hence between architects and engineers.

This type of methodologies are often valued for their intrinsic visual and intuitive nature. To this end, of particular importance are the core concepts of reciprocal form and force diagrams - geometrical elements of which represent structural members or their internal forces. These equip the design and analysis frameworks with a number of advantages such as: incorporation of structural performance in the early conceptual design stages; an interactive approach where designers have more leverage, design and analysis freedoms on the outcome of the model; structural morphogenesis opportunities by manipulating reciprocal diagrams; a new approach of design for structures which are by definition in static equilibrium (as opposed to conventional free-from design). This research paper will explore such structural morphogenesis capabilities for spatial trusses in static equilibrium based on the idea of reciprocal 4-polytopic stress functions.

2 Background

2.1 Graphic statics

The development of graphic statics can be attributed to the legacy of Da Vinci, Galilei, Newton (Zalewski and Allen 1998), Hooke, Poleni and Stevin (Heyman 1995) among others. Following Varignon's funicular polygon (Varignon 1725), Cremona, Cullman, Bow, Maxwell and Rankine contributed significantly to their development throughout the 19th century. For a detailed historical analysis the reader is pointed to Kurrer (2008). It should be highlighted that Maxwell is acknowledged as the originator of the concept of reciprocity between form and force diagrams (Charlton 1982; Kurrer 2008; Zalewski and Allen 1998) and the one who proposed a geometrical construction of these diagrams within the context of projective geometry.

Maxwell's graphical approach to the analysis of trusses (Maxwell 1864a, 1870) was heavily influenced by Chasles', Monge's and Poncelet's contribution to geometry at the time. In particular, from the pole and polar construction and the principle of duality, which expressed the reciprocity between form and force diagrams. Moreover, Maxwell synthesised the duality principle of projective geometry with Euler's work on polyhedral counting to develop a theory of reciprocal diagrams in statics. He made the profound observation that two-dimensional reciprocal diagrams obey the counting rules of polyhedra - when it comes to their constituent geometrical elements (points, edges, faces) - and that a 2-dimensional form diagram has a force reciprocal when it is a projection of a polyhedron (Maxwell 1864b). Furthermore, Maxwell was familiar with the work of Airy (Airy 1862) - and thus with the Airy stress function - which combined with his knowledge of Poncelet's and Monge's theory of polar figures (Charlton 1982) to develop a construction of reciprocal polyhedral stress functions the projections of which are reciprocal form and force diagrams (Maxwell 1870). This type of constructions did not find a widespread application among practitioners and researchers throughout the 20th century. One of the notable exceptions was in the field of theoretical mathematics, and in particular of rigidity theory, from the structural topology group at the University of Montreal (Crapo 1979; Crapo and Whiteley 1994). Moreover, Maxwell's constructions - and in particular his combination of the projective geometry duality along with a metric - were further developed from Strubecker (Strubecker 1962) in the context of isotropic geometry in 1960s.

The visual and intuitive appeal of graphic statics along with contemporary CAD developments and tne intrinsic interrelation between form and force resulted in numerous educational, computational and theoretical applications and advancements over the last two decades such as: engineering and architectural education (Allen and Zalewski 2010; Muttoni, A. 2011); design of compression-only or tension-only spatial funicular structures by means of the Thrust Network Analysis (TNA) (Block and Ochsendorf 2007); design of compression-only or tension-only plane-faced polyhedral structures based on subdivisions and manipulations of spatial force diagrams (Akbarzadeh et al. 2015); transformation of form diagrams in static equilibrium (Fivet 2016); design and analysis of spatial structures (D'Acunto et al. 2019); optimisation tools developed in applied research groups in industry (Beghini et al. 2014; Mazurek et al. 2016) and optimisation of grid-shells (Pellis and Pottmann 2018). It should be highlighted that the majority of current graphic statics approaches rely on the use of iterative algorithms and procedural reconstruction techniques and operate on a local node-by-node basis. Alternatively, direct global implementations include McRobie (2016); Konstantatou et al. (2018), which are grounded on the concept of higher dimensional reciprocal stress functions as a technique for generating pairs of reciprocal form and force diagrams.

2.2 Reciprocal diagrams and discrete stress functions

Reciprocal form and force diagrams, were developed as early as Hooke's time (17th century) for the safety assessment of masonry structures (Charlton 1982; Kurrer 2008). However, their in-depth conception and definition, has been attributed to Maxwell for the 2D case (Maxwell 1864b, 1870) and Rankine (Rankine 1864) for the 3D case. In the 2D case, form edges map to force edges and form nodes to closed force polygons. As a result, a 3D duality is obeyed between reciprocal geometrical elements. This 3D duality is in fact between the overarching reciprocal polyhedral stress functions which can be mapped to each other through a polar transformation, or polarity, using a paraboloid of revolution as Maxwell succinctly mentioned (Maxwell 1870). In the 3D case, form edges correspond this time to force faces, and nodes to closed polyhedral cells. As a result, a 4D duality is obeyed between the overarching 4-polytopic stress functions. Thus plane-faced, polyhedral, 3D trusses in static equilibrium are projections of 4-polytopic plane-faced stress functions. These were defined in equation form in Maxwell (1870) and in contemporary nomenclature are called Maxwell-Rankine stress functions (McRobie 2016).

Other properties of reciprocal form and force diagrams are: their interchangeability (either can be seen as the form or force) and the fact that there is no distinction between lines of action of the external forces and structural members (McRobie et al. 2016; Mitchell et al. 2016). The result of the latter is that external forces can be combined with the form diagram to an equivalent self-stress truss. Consequently, there is no distinction between these two cases (self-stressed, with external loading) and they are geometrically equivalent.

The frameworks we analyse here are self-stressed, pin-jointed structures. We will denote a 3D framework of a polyhedral structure $\mathbf{P}$ as a set of of vertices v, edges e, faces f and cells c: $\mathbf{P}(v,e,f,c)$ and its reciprocal to be $\mathbf{P}'(v',e',f',c')$. These can also be lifted in 4-dimensional space to a pair of reciprocal 4-polytopes. It should be noted that 4-polytopes are the equivalent of polyhedra in the 4-dimensional space where the cells lie on hyper-planes in the same way that faces lie on planes; hyper-planes constitute 3-dimensional subspaces in 4-dimensional space in the same way that planes constitute 2-dimensional sub-spaces in 3-dimensional space.

The correspondence between geometrical elements for 2D reciprocal diagrams is as follows: for a form diagram $\mathbf{F}(v,e,f)$ we have that for its reciprocal $\mathbf{F}'(v',e',f')$: $v = f', e = e', f = v'$. Moreover, a necessary condition for the existence of two reciprocal figures is that every e, e' belongs only to two polygons from f, f'.

Equivalently, the correspondence between geometrical elements for 3D plane-faced reciprocal diagrams is as follows: between form edges e and force faces f'. Moreover, points v are mapped to reciprocal cells c' indicating a 4D duality where $v = c', e = f', f = e', c = v'$ and thus $\mathbf{F}(v,e,f,c)$ maps to $\mathbf{F}'(v',e',f',c')$. Also, each face of the polyhedral diagram belongs only to two polyhedral cells, every line is the intersection of at least three faces and there are no free edges, faces, or points. This type of reciprocal geometrical construction was firstly described by Rankine (1864) and thus this type of spatial force reciprocals are called 'Rankine'.

2.3 Polarities and duality in the context of graphic statics

The principle of duality in 2D projective space means that any proposition that is true for points and lines can be dualised to an equivalently true proposition for lines and points (Cremona 1885). Thus this principle interchanges primitive geometrical elements of the projective plane. This duality can be extended to higher dimensions. For example, for 4D space, the interchangeable elements are the point, the line, the plane and the hyperplane.

The duality between reciprocal geometrical elements also applies to their connectivity matrices. For example, and for the 4D case, $C_p, C_l, C_{pl}, C_{h-pl}$ are the connectivity matrices between points, lines, planes and hyper-planes of a form diagram, and $C_{p'}, C_{l'}, C_{pl'}, C_{h-pl'}$ the corresponding connectivity matrices of its reciprocal. These matrices are symmetric and their entries are either 1, if two elements are adjacent (for instance, two points in C_p are connected from an edge or two planes in C_{pl} share an edge), or 0 if they are not.

Polarities are transformations, of degree two, which map every element in a space to another and are a useful tool for generating global force reciprocals of structures in static equilibrium as well as for transforming structures from one typology to another. Polarities in the context of contemporary graphic statics were introduced in Konstantatou et al. (2018) for the design and analysis of 3D structures. It should be noted that these constructions were mentioned in the context of rigidity theory in the 90s (Crapo and Whiteley 1994). Here we will outline only some useful equations which are necessary for this research. The reader is pointed to Konstantatou et al. (2018) for a more detailed account on this subject. Polarities, can be thought of as pairs of transformations L, L^{-1} that, in 3D space, map a plane π, defined in equation form as $Ax+By+Cz+D=0$ and described by the corresponding quadruples (A,B,C,D), to a point P described from the triple (x',y',z') and vice versa. Using, for example, a paraboloid of revolution with equation $x^2+y^2-2cz=0$ as the quadric of the polarity, for a point $P(x',y',z')$ and its polar plane π described by the equation $z=Ax+By+D$, we have that

$$L^{-1}(P)=\pi \tag{1}$$

$$L^{-1}(x',y',z')=(A,B,-1,D) \tag{2}$$

then for this particular quadric, the plane π is given by

$$x(x')+y(y')+z(-c)-cz'=0$$

which rearranges to

$$z=\frac{x'}{c}x+\frac{y'}{c}y-z' \tag{3}$$

which, when expressed in the (A,B,-1,D) form has

$$A=\frac{x'}{c}, \qquad B=\frac{y'}{c}, \qquad D=-z' \tag{4}$$

These equations can be readily inverted to give

$$x'=cA, \qquad y'=cB, \qquad z'=-D \tag{5}$$

which is thus the mapping L:

$$L(\pi)=P \tag{6}$$

$$L(A,B,-1,D)=(x',y',z')=(cA,cB,-D) \tag{7}$$

In the 4-dimensional space, a polar hyper-plane π defined in equation form as $Ax+By+Cz+Dw+E=0$ - and described by the corresponding quadruples (A,B,C,D,E) - is mapped to a point P described from the quadruple (x',y',z',w') (and vice versa) through transformations L, L^{-1}. Specifically, given a point P outside a hyper-quadric Γ (i.e. a 4-dimensional generalisation of a 3-dimensional quadric), the hyper-cone with vertex in P and tangent to Γ, intersects Γ in a quadric that lies on the hyper-plane π. Using, for example, a hyper-paraboloid of revolution with equation $2cw = x^2+y^2+z^2$ as the quadric of the polarity, for a point $P(x',y',z',w')$ and its polar plane π described by the equation $w = Ax+By+Cz+D$, we have that

$$L^{-1}(P) = \pi \tag{8}$$

$$L^{-1}(x',y',z',w') = (A,B,C,-1,E) \tag{9}$$

For this particular hyper-quadric, the hyper-plane π is given by

$$x(x')+y(y')+z(z')+w(-c)-cw' = 0$$

which rearranges to

$$w = \frac{x'}{c}x+\frac{y'}{c}y+\frac{z'}{c}z-w' \tag{10}$$

which, when expressed in the $(A,B,C,-1,E)$ form has

$$A = \frac{x'}{c}, \qquad B = \frac{y'}{c}, \qquad C = \frac{z'}{c}, \qquad E = -w' \tag{11}$$

These equations can be readily inverted to give

$$x' = cA, \qquad y' = cB, \qquad z' = cC, \qquad w' = -E \tag{12}$$

which is thus the mapping L:

$$L(\pi) = P \tag{13}$$

$$L(A,B,C,-1,E) = (x',y',z',w') = (cA,cB,cC,-E) \tag{14}$$

Following the above construction, it is possible to directly obtain force reciprocals when the form diagrams are projections of 4D stress functions $\mathbf{P}(v,e,f,c)$, $\mathbf{P}'(v',e',f',c')$. These are spatial polyhedral geometries with plane faces, each one of which belongs to exactly two cells.

3 Methods

The methodology followed here is based on the introduction of reciprocal stress functions which exist one dimension up from the corresponding reciprocal diagrams. That is, in 3-dimensional space for 2-dimensional cases and 4-dimensional space for 3-dimensional cases. This approach provides the agility to start from any one of the interlinked reciprocal objects (form, force, or their stress functions) while designing within equilibrium space. The main concept is that reciprocal stress functions can be mapped to each other and then subsequently projected one dimension down to produce reciprocal form and force diagrams (**fig. 1**). Thus, this process comprises a direct way to obtain global static equilibrium. The resulting method has no need for iterative convergence whilst the geometry of stress functions holds all the information of the structural behaviour of the corresponding truss: which members are in tension, which are in compression and by how much while providing possible design and analysis freedoms. This method is applicable to any 2D truss geometry and to any polyhedral 3D geometry which has plane faces, each one of which belongs to two cells. For these latter spatial cases, which are the focus of this research paper, the method is equally applicable to tension-only, compression-only, and tension-and-compression structures of any topology and, potentially intersecting, geometry. Moreover, it is applicable both to self-stressed and externally loaded structures.

The polarities graphic statics framework which was introduced previously has the following characteristics: all four reciprocal objects are interlinked and all of them can be altered while updating the rest, thus the designer has the option of choosing the starting point: form, force, or reciprocal stress functions; each one of these starting points has different capabilities but all result in statically equilibrated structures. Specifically, by starting with the force diagram as an input the designer can essentially design with the forces and observe interactively the form as an output of the process. Alternatively, by working with the force stress function, or by imposing the corresponding geometrical constraints on its projection (the force diagram), a structural morphogenesis method can be obtained where the resulting form diagram is guaranteed to be in static equilibrium. For the 3D case, the force diagram should be a projection of a 4-polytope. This latter condition is essentially what is already known in the literature as plane-faced (non-intersecting) Rankine reciprocals which produce compression- or tension-only polyhedral spatial structures (Akbarzadeh et al. 2015). This methodology includes subdivisions of the force diagrams which result in different topologies of the spatial truss. By using the polarities approach, this structural morphogenesis approach can be generalised for compression-and-tension structures, where the geometry of the faces (in terms of

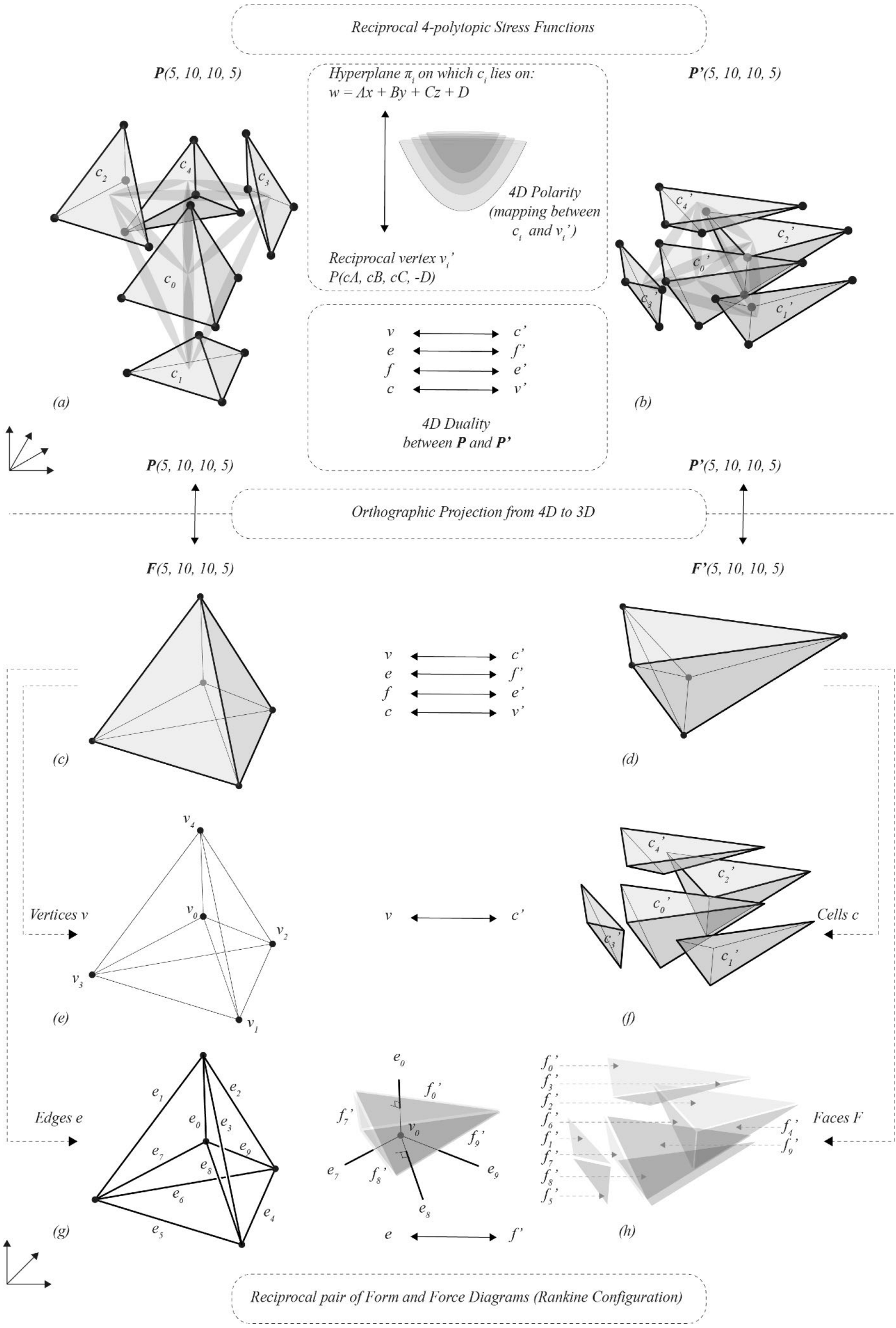

Figure 1: a, b: Pair of reciprocal 4-polytopic stress functions $\mathbf{P}, \mathbf{P}'$ which can be derived by mapping hyper-planes to reciprocal vertices through a 4D polarity. The resulting pair follows a 4D duality with respect to its constituent geometrical elements which is inherited to the pair of reciprocal form and force diagrams in 3D $\mathbf{F}, \mathbf{F}'$ since the latter can be obtained from the former through an orthographic projection from 4D to 3D. Adapted from Konstantatou et al. (2018).

intersections, or complexity) is not limited. This is because something wrapped and complicated in 3D can unravel and inflate in 4D. Moreover, by controlling the curvature of the stress function the designer can control the areas of tension and compression. Since form and force diagrams have an interchangeable character no special treatment is needed for any of these.

Alternatively, the designer can choose to work directly in the form space. In this case, by following the geometrical constraints imposed by the projective interlink of the form diagram and its corresponding stress function one can be sure that the designed form is structurally sound and not merely a free-form design with no structural performance considerations. What is more, even if the designer starts to draw a truss in a free-form manner then the resulting geometry can be *corrected* by ensuring that it is a projection of a higher dimensional stress function. For example, a polyhedral lifting can be performed on a 2D truss to check whether the latter is a projection of a plane-faced polyhedron. In this process, the location of a number of nodes can be corrected until the geometrical constraint is satisfied and the resulting truss is in static equilibrium. Moreover, the fact that these geometrical objects are interlinked through a direct transformation ensures global static equilibrium in every step and there is no need for iterative node-by-node reconstructions and imposed approximations between the form and force reciprocals. As a result, stress functions can be used as a tool for structural morphogenesis of forms in static equilibrium. Here we will design spatial trusses in the form of exoskeletons/ mega trusses of high-rise towers.

The computational tools developed with respect to the above methodology were implemented in the Grasshopper platform of the Rhino CAD software (**fig. 2**). In particular, custom scripts were developed in the Python programming language which would: *a*) lift the cells of the spatial form diagram to their corresponding 4-dimensional hyper-planes; *b*) map these following a direct mathematical operation to their respective reciprocal force vertices in 4D; *c*) create all the other force geometrical elements (edges, faces and cells), via 4D duality, by following the connectivity of the initial form diagram; and *d*) Project the resulting reciprocal pair of 4-polytopic stress functions orthogonally to the 3rd dimension to obtain the reciprocal pair of spatial form and force diagrams. For the algebraic solver developed with regards to *a*, the Math.NET Numerics library was used.

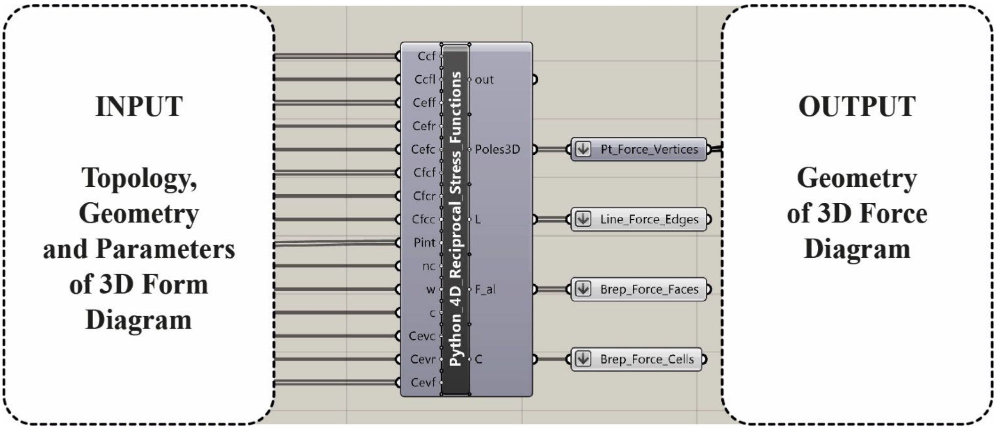

Figure 2: The computational tool implemented in Python within the Grasshopper/ Rhino CAD environment.

4 Results and Reflection

This methodology can be seen as a direct application of Maxwell's Figure IV (Maxwell 1864a) in the 4th dimension. Specifically, in (fig. 3) we see how two planes intersect in 3-space to a line w and at the same time define a pyramid with a flat top - the projection of which is a truss in static equilibrium. In fact, the line *w* signifies the relative position of these two planes and is thus a design freedom with regards to the design of the 2D truss. Equivalently, if this construction is translated one dimension up in 4-space (fig. 3) then two hyperplanes will intersect to a plane w (which now signifies a design freedom), these two hyperplanes can be used to define a plane-faced 4D pyramid, the projection of which is thus guaranteed to be in static equilibrium. The lines of action of the external forces are added to the structural members resulting to an equivalent self-stressed truss which can be then be analysed. As outlined above, the designer can start from the force diagram (fig. 4) - the topology and geometry of which will define the resulting spatial truss. Alternatively, the designer can choose to work straight in the form space. This particular geometry is essentially two interconnected envelopes.

By following this methodology and working solely in the form space a wide range of typologies for towers in static equilibrium can be generated as in (fig. 6). These geometries are self-stressed spatial systems comprising interconnected polyhedral truss envelopes. Each one of these systems will have a Rankine force reciprocal (fig. 4) which visualises the state of self-stress of the tower. The number of interconnected polyhedral envelopes can be decided by the designer based on the architectural brief. The geometry of (fig. 4) was selected for developing a physical model of this tower typology. In collaboration with Dr. Masoud Akbarzadeh, Dr. Andrei Nejur and PhD students from the Polyhedral Structures Lab at UPenn,

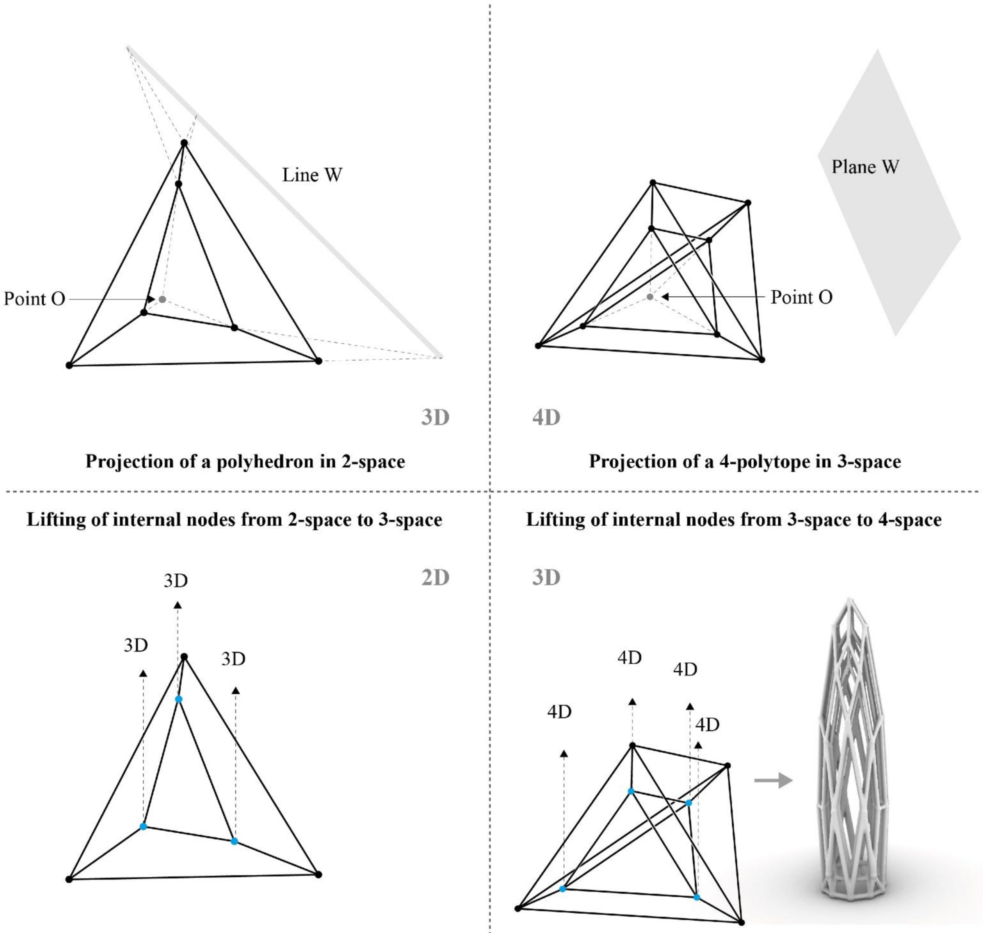

Figure 3: Left-Top: Maxwell's Figure IV (Maxwell 1864a) is a projection of a polyhedron where the position of line w signifies a design freedom; Left-Bottom: Polyhedral lifting of Maxwell's Figure IV; Right-Top: The 3D generalisation of Maxwell's Figure IV is a projection of a 4-polytope where the position of plane w signifies a design freedom; Right-Bottom: 4-polytopic lifting of the form diagram, showing the geometrical principles resulting in self-stressed spatial trusses in static equilibrium such as towers.

a computational and physical model was developed which comprised structural members of varying diameters and custom-made joints which connected to the edges. In particular, the structural members were timber rods whereas the joints were 3D-printed with flexible plastic material. The glass facade panels were laser-cut from perspex material and attached to the timber rods via 3D-printed rings. The finished model (**fig. 5**) is 1m heigh and has been exhibited at UPenn.

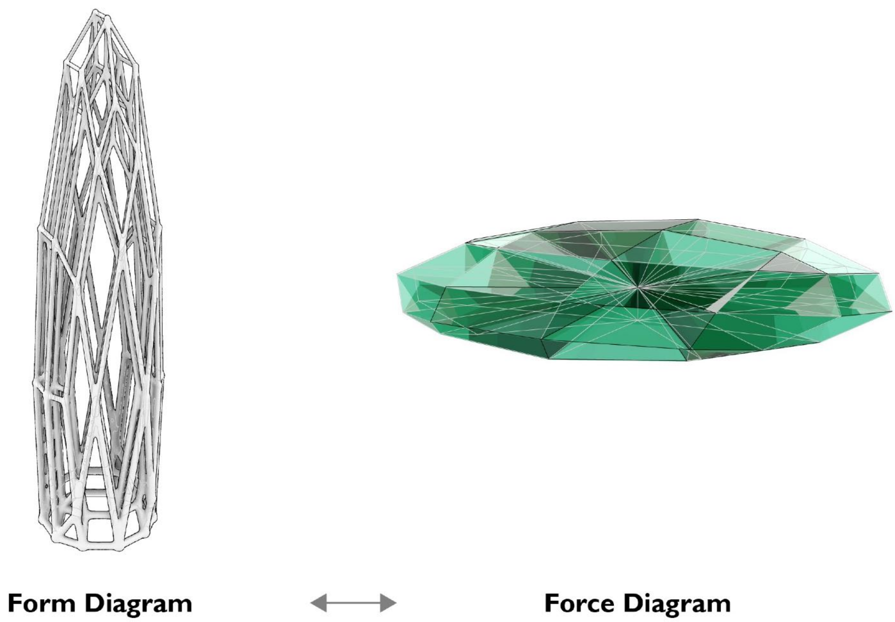

Figure 4: Self-stressed mega truss of a tower in static equilibrium as a form diagram and its reciprocal Rankine 3D force diagram.

5 Conclusion

In this research paper we discussed how Maxwell's graphic statics approach, comprising reciprocal polyhedral Airy stress functions and their interlinked form and force diagrams, can be generalised in the 4^{th} dimension and applied to the structural morphogenesis of spatial trusses for towers in static equilibrium. This approach can produce a plethora of compression-and-tension typologies which are by definition within equilibrium space whilst their Rankine force reciprocals can be directly produced and visualised. We synthesised the direct mathematical description with CAD capabilities to develop tools which enable the designers to analyse and design this type of structures in a bi-directional way, namely, starting from either the form or the force - or in fact from the higher dimensional stress functions - while visualising the design freedoms in terms of geometrical objects.

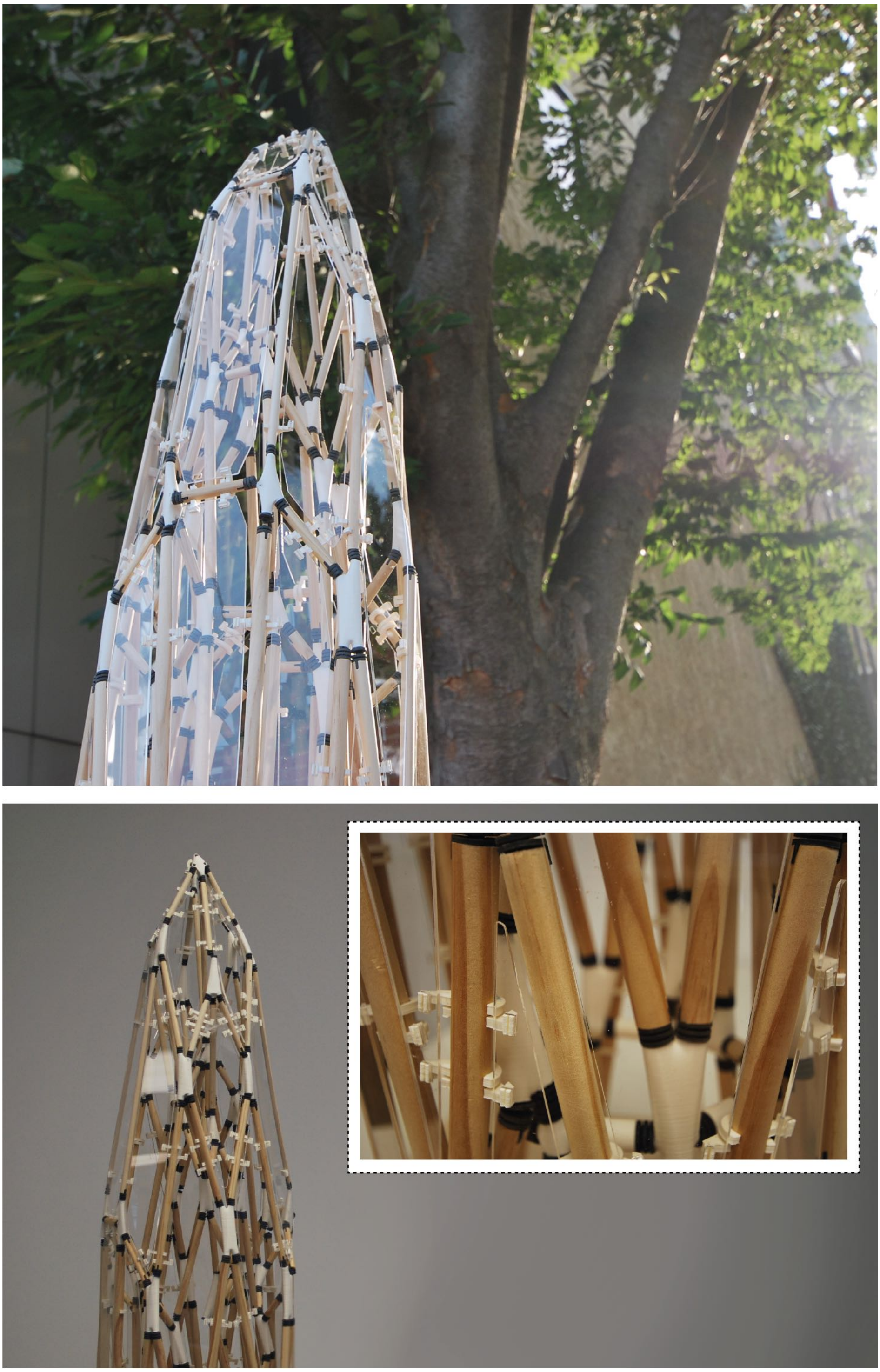

Figure 5: Various typologies and heights of trusses of towers in static equilibrium.

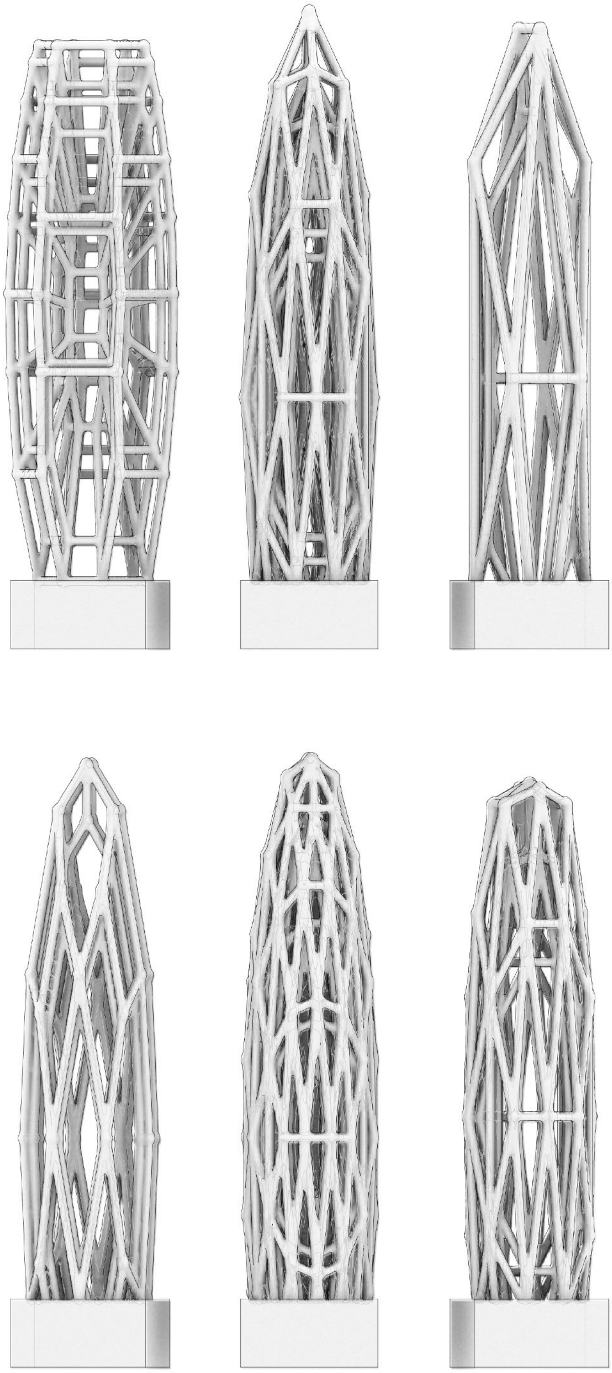

Figure 6: CAD and CAM produced physical model of the 4-polytopic tower typology.

References

Airy, G. B. (1862). On the Strains in the Interior of Beams. *Philosophical Transactions of the Royal Society of London 153*, 49–79.

Akbarzadeh, M., T. Van Mele, and P. Block (2015). On the equilibrium of funicular polyhedral frames and convex polyhedral force diagrams. *Computer-Aided Design 63*(Supplement C), 118–128.

Allen, E. and W. Zalewski (2010). *Forms and Forces*. New Jersey: John Wiley & Sons.

Beghini, L., J. Carrion, A. Beghini, A. Mazurek, and W. Baker (2014). Structural optimization using graphic statics. *Structural and Multidisciplinary Optimization 49*(3), 351–366.

Block, P. and J. Ochsendorf (2007). Thrust Network Analysis: a new methodology for three-dimensional equilibrium. *Journal of the International Association for Shell and Spatial Structures 48*(3).

Charlton, T. (1982). *A History of Theory of Structures in the 19th century*. Cambridge: Cambridge University Press.

Crapo, H. (1979). Structural Rigidity. *Structural Topology 1*, 26–45.

Crapo, H. and W. Whiteley (1994). 3-Stresses in 3-Space and Projections of Polyhedral 3-surfaces: Reciprocals, Liftings and Parallel Configurations. *Preprint*.

Cremona, L. (1885). *Elements of Projective Geometry*. Oxford: Clarendon Press.

D'Acunto, P., J.-P. Jasienski, P. O. Ohlbrock, C. Fivet, J. Schwartz, and D. Zastavni (2019). Vector-based 3d graphic statics: A framework for the design of spatial structures based on the relation between form and forces. *International Journal of Solids and Structures 167*, 58–70.

Fivet, C. (2016). Projective transformations of structural equilibrium. *International Journal of Space Structures 31*, 135–146.

Heyman, J. (1995). *The Stone Skeleton: Structural Engineering of Masonry Architecture*. Cambridge: Cambridge University Press.

Konstantatou, M., P. D'Acunto, and A. McRobie (2018). Polarities in structural analysis and design: n-dimensional graphic statics and structural transformations. *International Journal of Solids and Structures 152-153*, 272–293.

Kurrer, K.-E. (2008). *The History of the Theory of Structures, From Arch Analysis* *to Computational Mechanics*. Germany: Ernst & Sohn.

Maxwell, J. (1864a). On Reciprocal Figures and Diagrams of Forces. *Philosophical Magazine and Journal of Science 26*, 250–261.

Maxwell, J. (1864b). On the calculation of equilibrium and the stiffness of frames. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science XXVII*, 294–299.

Maxwell, J. (1870). On Reciprocal Figures, Frames, and Diagrams of Forces. *Transactions of the Royal Society of Edinburgh 7*, 160–208.

Mazurek, A., A. Beghini, J. Carrion, and W. F. Baker (2016). Minimum weight layouts of spanning structures obtained using graphic statics. *International Journal of Space Structures 31*(2-4), 112–120.

McRobie, A. (2016). Maxwell and Rankine reciprocal diagrams via Minkowski sums for two-dimensional and three-dimensional trusses under load. *International Journal of Space Structures 31*(2-4), 203–216.

McRobie, A., W. Baker, T. Mitchell, and M. Konstantatou (2016). Mechanisms and states of self-stress of planar trusses using graphic statics, Part II: Applications and extensions. *31*(2), 102–111.

Mitchell, T., W. Baker, A. McRobie, and A. Mazurek (2016). Mechanisms and states of self-stress of planar trusses using graphic statics, part I: The fundamental theorem of linear algebra and the Airy stress function. *International Journal of Space Structures 31*(2-4), 85–101.

Muttoni, A. (2011). *The Art of Structures: Introduction to the Functioning of Structures in Architecture*. EPFL Press.

Pellis, D. and H. Pottmann (2018). Aligning principal stress and curvature directions. In *Advances in Architectural Geometry*, Gothenburg, Sweden.

Rankine, M. (1864). Principle of the equilibrium of polyhedral frames. *Philosophical Magazine 27*, 92.

Strubecker, K. (1962). Airy'sche spannungsfunktion und isotrope differentialgeometrie. *American Journal of Physics 78*, 189–198.

Varignon, P. (1725). *Nouvelle Mecanique ou Statique*. Paris: Claude Jombert.

Zalewski, W. and E. Allen (1998). *Shaping Structures, Statics*. Unites States of America: John Wiley & Sons, INC.